

# An extremal problem for a class of entire functions of exponential type

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## Abstract

Let  $f$  be an entire function of exponential type whose indicator diagram is contained in the segment  $[-i\sigma, i\sigma]$ ,  $\sigma > 0$ . Then the upper density of zeros of  $f$  is at most  $c\sigma$  where  $c \approx 1.508879$  is the positive solution of the equation

$$\log(\sqrt{c^2 + 1} + c) = \sqrt{1 + c^{-2}}.$$

This estimate is exact.

We consider the class  $E_\sigma$ ,  $\sigma > 0$  of entire functions of exponential type whose indicator diagram is contained in a segment  $[-i\sigma, i\sigma]$ , which means that

$$h(\theta) := \limsup_{r \rightarrow +\infty} \frac{\log |f(re^{i\theta})|}{r} \leq \sigma |\sin \theta|, \quad |\theta| \leq \pi. \quad (1)$$

An alternative characterization of such functions follows from a theorem of Pólya [6]:

$$f(z) = \frac{1}{2\pi} \int_{\gamma} F(\zeta) e^{-i\zeta z} d\zeta,$$

where  $F$  is an analytic function in  $\overline{\mathbf{C}} \setminus [-\sigma, \sigma]$ ,  $F(\infty) = 0$ , and  $\gamma$  is a closed contour going once around the segment  $[-\sigma, \sigma]$ . In other words, the class of

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entire functions satisfying (1) consists of Fourier transforms of hyperfunctions supported by  $[-\sigma, \sigma]$ , see, for example, [2] and [3].

Let  $n(r)$  be the number of zeros of  $f$  in the disc  $\{z : |z| \leq r\}$ , counting multiplicity. We are interested in the *upper density*

$$D = \limsup_{r \rightarrow \infty} \frac{n(r)}{r}. \quad (2)$$

If  $f$  satisfies the additional condition

$$\int_{-\infty}^{\infty} \frac{\log^+ |f(x)|}{1+x^2} dx < \infty, \quad (3)$$

then the limit (density) in (2) exists and equals  $(2\pi)^{-1} \int_{-\pi}^{\pi} h(\theta) d\theta$ . For example, if  $f(z) = \sin \sigma z$ , then  $f \in E_\sigma$  and  $D = 2\sigma/\pi \approx 0.6366\sigma$ . The existence of the limit follows from a theorem of Levinson [5, 6]. Much more precise information about  $n(r)$  under the condition (3) is contained in the theorem of Beurling and Malliavin [1].

In the general case, the density might not exist as was shown by examples in [4, 10]. Moreover, it is possible that  $D > 2\sigma/\pi$ , see [2]. An easy estimate using Jensen's formula gives  $D \leq 2e\sigma/\pi \approx 1.7305\sigma$ . This estimate is exact in the larger class of entire functions satisfying the condition  $h(\theta) \leq \sigma$ , but it is not exact in  $E_\sigma$ .

In this paper we find the best possible upper estimate for the upper density of zeros of functions in  $E_\sigma$ .

**Theorem.** *The upper density of zeros of a function  $f \in E_\sigma$  does not exceed  $c\sigma$  where  $c \approx 1.508879$  is the unique solution of the equation*

$$\log(\sqrt{c^2 + 1} + c) = \sqrt{1 + c^{-2}}, \quad \text{on } (0, +\infty). \quad (4)$$

*For every  $\sigma > 0$  there exist entire functions  $f \in E_\sigma$  such that  $D = c\sigma$ .*

*Proof.* Without loss of generality we assume that  $\sigma = 1$ . Moreover, it is enough to consider only even functions. To make a function  $f$  even we replace it by  $f(z)f(-z)$ , which results in multiplication of both the indicator  $h$  and the upper density  $D$  by the same factor of 2.

Let  $t_n \rightarrow +\infty$  be such sequence that  $\lim n(t_n)/t_n = D$ . Consider the sequence of subharmonic functions  $v_n(z) = t_n^{-1} \log |f(t_n z)|$ . Compactness

Principle for subharmonic functions [3, Theorem 4.1.9] implies that one can choose a subsequence that converges in  $\mathscr{D}'$  (Schwartz's distributions). The limit function  $v$  is subharmonic in the plane, and satisfies

$$v(z) \leq |\operatorname{Im} z|, \quad z \in \mathbf{C}, \quad \text{and} \quad v(0) = 0. \quad (5)$$

Let  $\mu$  be the Riesz measure of this function. We have to show that

$$\mu(\{z : |z| \leq 1\}) \leq c. \quad (6)$$

First we reduce the problem to the case that the Riesz measure  $\mu$  is supported by the real line. We have

$$v(z) = \frac{1}{2} \int \log \left| 1 - \frac{z^2}{\zeta^2} \right| d\mu_\zeta.$$

Let us compare this with

$$v^*(z) = \frac{1}{2} \int_0^\infty \log \left| 1 - \frac{z^2}{t^2} \right| d\mu_t^*,$$

where  $\mu^*$  is the radial projection of the measure  $\mu$ : it is supported on  $[0, +\infty)$  and  $\mu^*(a, b) = \mu(\{z : a < |z| < b\})$ ,  $0 \leq a < b$ . It is easy to see that

$$v^*(z) \leq \sigma' |\operatorname{Im} z|, \quad z \in \mathbf{C}, \quad \text{and} \quad v^*(0) = 0 \quad (7)$$

with some  $\sigma' > 0$ . We claim that one can choose  $\sigma' \leq 1$  in (7). Let  $\sigma'$  be the smallest number for which (7) holds. Then, by the subharmonic version of the theorem of Levinson mentioned above (see, for example, [9]), the limit

$$\lim_{r \rightarrow \infty} r^{-1} v^*(rz) = \sigma' |\operatorname{Im} z|$$

exists in  $\mathscr{D}'$  and thus

$$\lim_{r \rightarrow \infty} \frac{1}{r} \int_0^r \frac{n_{v^*}(t)}{t} dt = \lim_{r \rightarrow \infty} \frac{1}{2\pi r} \int_{-\pi}^\pi v^*(re^{i\theta}) d\theta = 2\sigma'/\pi,$$

where

$$n_{v^*}(r) = \mu^*[0, r] = \mu\{z : |z| \leq r\}. \quad (8)$$

Similar limits exist for  $v$ , and we have  $n_v = n_{v^*}$ , from which we conclude that  $\sigma' \leq 1$ .

From now on we assume that  $v$  is harmonic in the upper and lower half-planes, and that

$$v(iy) \sim y, \quad y \rightarrow +\infty. \quad (9)$$

Let  $u$  be the harmonic function in the upper half-plane such that  $\phi = u + iv$  is analytic, and  $\phi(0) = 0$ . Then  $\phi$  is a conformal map of the upper half-plane onto some region  $G$  of the form

$$G = \{x + iy : y > g(x)\}, \quad (10)$$

where  $g$  is an even upper semi-continuous function,  $g(0) = 0$ . Moreover,

$$\phi(iy) \sim iy, \quad \text{as } y \rightarrow +\infty, \quad (11)$$

which follows from (9), and

$$\phi(-\bar{z}) = -\overline{\phi(z)}, \quad (12)$$

because both the region  $G$  and the normalization of  $\phi$  are symmetric with respect to the imaginary axis. Finally we have

$$\mu([0, x]) = \frac{2}{\pi} u(x). \quad (13)$$

For all these facts we refer to [7].

*Remark.* The function  $\operatorname{Re} \phi(x) = u(x)$  might be discontinuous for  $x \in \mathbf{R}$ . We agree to understand  $u(x)$  as the limit from the right  $u(x+0)$  which always exists since  $u$  is increasing.

Inequality (5) implies that  $v(x) \leq 0$ , thus  $g(x) \leq 0$ , in other words,  $G$  contains the upper half-plane.

Thus we obtain the following extremal problem: *Among all univalent analytic functions  $\phi$  satisfying (12) and mapping the upper half-plane onto regions of the form (10) with  $g \leq 0$ ,  $g(0) = 0$  and satisfying  $\phi(0) = 0$  and (11), maximize  $\operatorname{Re} \phi(1)$ .*

We claim that the extremal function  $g$  for this problem is

$$g_0(x) = \begin{cases} -\infty, & 0 < |x| < \pi c/2, \\ 0, & \text{otherwise,} \end{cases}$$

where  $c > 1$  is the solution of equation (4). The corresponding region is shown in Fig. 1. For the extremal function we have  $\phi_0(1) = \pi c/2 - i\infty$ .

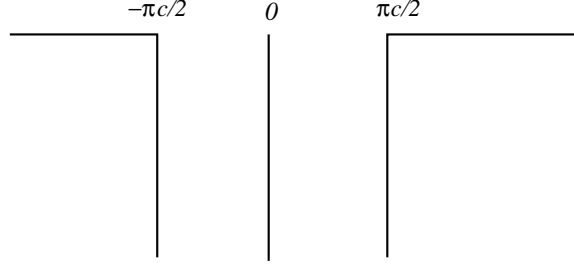


Fig. 1. Extremal region.

To prove the claim, we first notice that for a given  $G$  the mapping function is uniquely defined. Let  $a = \phi(1)$ , and  $b = \operatorname{Re} a$ . Next we show that making  $g$  smaller on the interval  $(0, b)$  results in increasing  $\operatorname{Re} \phi(1)$  and making  $g$  larger on the interval  $(b, +\infty)$  also results in increasing  $\operatorname{Re} \phi(1)$ . The proofs of both statements are similar. Suppose that  $g_1 \leq g$ ,  $g_1 \neq g$ , and  $g_1(x) = g(x)$  outside of the two intervals  $p < |x| < q$ , where  $0 < p < q < b$ . Let  $G_1$  be the region above the graph of  $g_1$ , and  $\phi_1$  the corresponding mapping function normalized in the same way as  $g$ . Then  $G \subset G_1$ , and the conformal map  $\phi_1^{-1} \circ \phi$  is defined in the upper half-plane and maps it into itself. We have

$$\phi_1^{-1} \circ \phi(x) = x + 2x \int_0^\infty \frac{w(t)}{t^2 - x^2} dt,$$

where  $w \neq 0$  is a non-negative function supported on some interval inside  $(0, 1)$ . Putting  $x = 1$  we obtain

$$\phi_1^{-1}(a) = 1 + 2 \int_0^\infty \frac{w(t)}{t^2 - 1} dt,$$

so  $\phi_1^{-1}(a) < 1$ , that is  $\operatorname{Re} \phi_1(1) > b$ . This proves our claim.

It remains to compute the constant  $b$  in the extremal domain. We recall that  $\phi_0(1) = b - i\infty$  and assume that  $b = \phi_0(k)$  for some  $k > 1$ . Here  $\phi_0$  is the extremal mapping function. Then by the Schwarz–Christoffel formula we have

$$\phi_0(z) = \frac{1}{2} \int_0^{z^2} \frac{\sqrt{\zeta - k^2}}{\zeta - 1} d\zeta. \quad (14)$$

To find  $k$ , we use the condition that

$$\operatorname{Im} p.v. \int_0^{k^2} \frac{\sqrt{\zeta - k^2}}{\zeta - 1} d\zeta = 0.$$

Denoting  $c = \sqrt{k^2 - 1}$  and evaluating the integral, we obtain

$$\log(\sqrt{c^2 + 1} + c) = \sqrt{1 + c^{-2}}.$$

Finally the jump of the real part of the integral in (14) occurs at the point 1 and has magnitude  $\pi\sqrt{k^2 - 1} = \pi c$ . This completes the proof of the upper estimate in Theorem 1.

To construct an example showing that this estimate can be attained, we follow the construction in [2, Sect.9-10]. The role of the subharmonic function  $u_1$  there is played now by our extremal function  $v_0 = \text{Im } \phi_0$ .

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